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Exact thresholds and instanton effects in $D = 3$ string theories

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Abstract: Three-dimensional string theories with 16 supersymmetries are believed to possess a non-perturbative U-duality symmetry $\mathrm{SO}(8, 24, \mathbb{Z})$. By covariantizing the heterotic one-loop amplitude under U-duality, we propose an exact expression for the $(\partial\phi)^4$ amplitude, that reproduces known perturbative limits. The weak-coupling expansion in either of the heterotic, type-II or type-I descriptions exhibits the well-known instanton effects, plus new contributions peculiar to three-dimensional theories, including the Kaluza-Klein 5-brane, for which we extract the summation measure. This letter is a post-scriptum to hep-th/0001083.

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1. Introduction

In a recent article [1], we have submitted the conjectured heterotic-type-II duality to its most detailed test to date, by showing the identity of the heterotic one-loop-exact four-gauge-boson F^4 threshold in six dimensions with the tree-level amplitude in type IIA compactified on K_3. This test was carried out at the $\mathbb{Z}_2$ orbifold point of K_3, and took several non-trivial facts to succeed, including the invariance under triality of automorphic forms of the T-duality group SO$(4, 4, \mathbb{Z})$ [2]. As a spin-off of this analysis, we obtained exact amplitudes in type I', F-theory and M-theory on K_3 at a singular point, as well as in type II compactified on $K_3 \times \mathbb{T}^d$ at the orbifold point, and extracted the contributions from D-brane instantons (for $d = 1, 2$) and for the first time NS5-brane instantons ($d = 2$).

In this letter, we extend our analysis to the case of the heterotic string compactified on $\mathbb{T}^7$ to $D = 3$ dimensions, or equivalently type-II string compactified on $K_3 \times \mathbb{T}^3$. This generalization is of interest for several reasons. Firstly, three-dimensional gravity theories have an interesting infrared behavior, due to the deficit angle created by massive particles, which allows to have a supersymmetric vacuum without Bose-Fermi degeneracy [3] (see also [4]). The possibility of dynamically generating a fourth dimension may then offer a solution to the cosmological constant problem [5]. Three-dimensional gauge theories may also have non-trivial infrared fixed points [6] which are far from being understood. Secondly, in contrast to $D > 3$, the perturbative moduli unify with the heterotic dilaton and the dualized gauge fields into a simple symmetric manifold SO$(8) \times$ SO$(24) \backslash$ SO$(8, 24)$ [7], identified under the conjectured

U-duality symmetry $SO(8,24,\mathbb{Z})$ [8]. Such a unification also happens in the case of M-theory compactified on a torus $\mathbb{T}^d$ (see [9] for a review) where the use of automorphic forms has proven efficient, and in type IIB on K_3, where the U-duality $SO(5,21,\mathbb{Z})$ symmetry has been put to little use (see however [1] for U-duality invariant $t_{12}H^4$ couplings, obtained by duality from the heterotic string on $\mathbb{T}^5$). Because of the unification of the dilaton with other moduli, any U-duality invariant amplitude is thus bound to contain instanton corrections along with the perturbative contributions, and hence to give us information in particular about the elusive heterotic instantons. Thirdly, string theories in three dimensions possess new instanton configurations that were not present in $D>3$, namely the Kaluza-Klein 5-brane instantons. These instantons are obtained by tensoring a Taub-NUT gravitational instanton asymptotic to $\mathbb{R}^3\times S^1$ with a flat $\mathbb{T}^6$, where $\mathbb{T}^7=S^1\times\mathbb{T}^6$. They are the ten-dimensional euclidean version [10] of the Kaluza-Klein monopoles introduced in [11]. Their worldvolume action has been constructed in [12], and their dynamics analyzed in [13]. Finally, $D=3$ is just one step away from $D=2$, where the U-duality group becomes infinite-dimensional, and predicts an infinite set of particles with exotic dependence $1/g_s^{n+3}$ on the coupling [14, 9]. Similar states already appear as particles in $D=3$ with mass $1/g_s^3$. Instantons in $D=3$ are however free of these infrared problems, and the study of exact amplitudes in $D=3$ may shed light on the non-perturbative spectrum.

The plan and main results of this paper are as follows. In section 2 we recall the reader about the way $SO(8,24,\mathbb{Z})$ duality arises in the heterotic string on $\mathbb{T}^7$, and how it manifests itself on the type-IIA/$K_3\times\mathbb{T}_3$ side. In particular, we show how the type-II $\mathbb{T}^3$ moduli are related by triality to the ones on the heterotic side, in much the same way as the type-II K_3 moduli are related by triality to the heterotic $\mathbb{T}^4$ moduli at the orbifold point [1, 15]. In section 3, we propose a U-duality invariant completion of the heterotic one-loop F^4 amplitude, which also reproduces the type II tree-level result at the orbifold point. From the non-perturbative expansion on the heterotic side we are able to identify new heterotic 5-brane and KK5-brane instanton effects. The corresponding expansion on the type-II side reveals, along with the Dp-brane and NS5-brane instanton effects found in [1], D6-brane and new KK5-brane instanton contributions as well. Finally, we comment on the type-I dual theory at the orbifold point, for which, besides the known perturbative terms [16] and D1-instantons [17, 18, 19], we identify D5-brane and KK5-brane instanton contributions.

2. U-duality and heterotic-type-II triality in $D=3$

In this section, we give a brief survey of the appearance of the $SO(8,24,\mathbb{Z})$ U-duality symmetry in heterotic on $\mathbb{T}^7$ or type II on $K_3\times\mathbb{T}^3$. For simplicity, we consider mainly compactifications on square tori without gauge background, even though the full duality becomes apparent only when including the Wilson line and vielbein moduli on the heterotic side, and the Ramond backgrounds on the type-II side.

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2.1 $\mathrm{SO}(8,24)$ invariance in heterotic on $\mathbb{T}^7$

The heterotic string compactified on $\mathbb{T}^7$ possesses the well-known T-duality invariance $\mathrm{SO}(7,23,\mathbb{Z})$. The scalars fields coming from the reduction of the ten-dimensional metric, B-field and U(1) gauge fields take value in a coset $[\mathrm{SO}(7)\times\mathrm{SO}(23)]\backslash\mathrm{SO}(7,23)$, parameterized as usual by a symmetric matrix $M_{7,23}$. The three-dimensional coupling constant $1/g_{3\mathrm{H}}^2 = V_7/(g_{\mathrm{H}}^2 l_{\mathrm{H}}^7)$ takes value in a separate $\mathbb{R}^+$ factor, inert under T-duality. In addition, there are $7+23$ U(1) gauge fields, which in three dimensions can be dualized into as many scalars. This is achieved by adding a Lagrange multiplier term $\phi_a dF^a$ to the gauge kinetic term $(l_{\mathrm{H}}/g_{3\mathrm{H}}^2)F^a(M_{7,23})^{-1}_{ab}F^b$ and integrating out the field strength F^a, which yields

$$*F^a = \frac{g_{3\mathrm{H}}^2}{l_{\mathrm{H}}} M_{7,23}^{ab} d\phi_b\,. \tag{2.1}$$

Together with the dilaton and the $\mathrm{SO}(7,23)$ moduli, these scalars span an $[\mathrm{SO}(8)\times\mathrm{SO}(24)]\backslash\mathrm{SO}(8,24)$ symmetric space, which encompasses all bosonic fields except the graviton.

Another way of getting to this result is to first compactify the heterotic string on $\mathbb{T}^6$ to four dimensions, and then further to $D=3$. In $D=4$, the moduli space has two factors, one is the usual $[\mathrm{SO}(6)\times\mathrm{SO}(22)]\backslash\mathrm{SO}(6,22)$ factor, acted upon by $\mathrm{SO}(6,22,\mathbb{Z})$ T-duality, and the second is $\mathrm{U}(1)\backslash\mathrm{Sl}(2)$, parameterized by the complex modulus $S = B + i/g_{4\mathrm{H}}^2$, where $1/g_{4\mathrm{H}}^2 = V_6/(g_{\mathrm{H}}^2 l_{\mathrm{H}}^6)$ is the four-dimensional string coupling constant and B the scalar dual to the Neveu-Schwarz two-form in four dimensions. The action of $\mathrm{Sl}(2,\mathbb{Z})$ on S is conjectured to be a non-perturbative symmetry of the heterotic string compactified on $\mathbb{T}^6$. Upon further compactification to $D=3$, the radius R_7, the Wilson lines of the 6+22 gauge fields and their scalar dual in $D=3$ make up again the $\mathrm{SO}(8,24)$ coset space.

This last point of view allows to easily identify the subgroup of $\mathrm{SO}(8,24)$ which remains as a quantum symmetry. Indeed, the action of the Weyl generator $g_{4\mathrm{H}} \to 1/g_{4\mathrm{H}}, l_{\mathrm{H}} \to g_{4\mathrm{H}}^2 l_{\mathrm{H}}$ of the $\mathrm{Sl}(2,\mathbb{Z})$ S-duality group translates in terms of the three-dimensional variables into the exchange of $1/g_{3\mathrm{H}}^2$ and R_7/l_{H}. Using the Weyl group of $\mathrm{Sl}(7)\subset\mathrm{SO}(7,23)$, we can transform $1/g_{3\mathrm{H}}^2$ into R_i/l_{H} for any radius of $\mathbb{T}^7$. This implies that we can think of the $\mathrm{SO}(8,24)$ scalars as the moduli of an heterotic compactification on $\mathbb{T}^7\times S^1$, where the radius of S_1 is given by

$$\frac{R_8}{l_{\mathrm{H}}} = \frac{1}{g_{3\mathrm{H}}^2}\,. \tag{2.2}$$

This compact circle therefore appears as a dynamically generated dimension decompactifying at weak coupling.[1] Note that this is *not* the usual M-theory direction, whose radius is instead $R_{11} = g_{\mathrm{H}} l_{\mathrm{H}}$. In particular, U-duality implies that there should be enhanced gauge symmetry at $R_8 = l_{\mathrm{H}}$, i.e. $g_{3\mathrm{H}} = 1$.

[1] It may seem more sensible to define the T-dual radius $\hat{R}_8/l_{\mathrm{H}} = g_{3\mathrm{H}}^2$ which decompactifies at strong coupling, but the limits $R\to 0$ and $R\to\infty$ are equivalent in the heterotic string.

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We have therefore identified the dilatonic moduli parameterizing the abelian part of $\mathrm{SO}(8,24)$,

$$x_{i=1,\dots,8} = \left(\frac{R_1}{l_{\mathrm{H}}}, \dots, \frac{R_7}{l_{\mathrm{H}}}, \frac{1}{g_{3\mathrm{H}}^2}\right) . \tag{2.3}$$

Including the 30 scalars ϕ_a of (2.1), the vielbein parameterizing the $\mathrm{SO}(8) \times \mathrm{SO}(24) \setminus \mathrm{SO}(8,24)$ coset can be chosen as

$$e_{8,24} = \begin{pmatrix} g_{3\mathrm{H}}^2 & & \\ & e_{7,23} & \\ & & g_{3\mathrm{H}}^{-2} \end{pmatrix} \cdot \begin{pmatrix} 1 & \phi & -\phi\eta_{7,23}\phi^t/2 \\ & 1_{30} & -\eta_{7,23}\phi^t \\ & & 1 \end{pmatrix} , \qquad e_{8,24}^t \eta_{8,24} e_{8,24} = \eta_{8,24} \tag{2.4}$$

where $e_{7,23}$ is the vielbein of the perturbative $\mathrm{SO}(7,23)$ moduli and we have defined

$$\eta_{d,d+16} = \begin{pmatrix} & & 1_d \\ & 1_{16} & \\ 1_d & & \end{pmatrix} . \tag{2.5}$$

Having in mind the decompactification of our forthcoming three-dimensional results to $D = 4$, we now discuss in some more detail the embedding of $\mathrm{SO}(6,22)$ into $\mathrm{SO}(8,24)$. The 7th direction taking from to $D = 4$ to $D = 3$ and the non-perturbative 8th direction form a dynamically generated torus $\mathbb{T}^2$ with complex structure and Kähler moduli

$$U = B + i\frac{R_8}{R_7} = S , \qquad T = A + i\frac{R_7 R_8}{l_{\mathrm{H}}^2} = A + i\frac{R_7^2 V_6}{g_{\mathrm{H}}^2 l_{\mathrm{H}}^8} , \tag{2.6}$$

where we recognize the complex structure U as the S modulus of the four-dimensional theory, and the imaginary part of the Kähler modulus T as the action of the Kaluza-Klein monopole on $\mathbb{T}^7$. The axionic part A is the dual of the Kaluza-Klein gauge field arising in the reduction of the metric from $D = 4$ to $D = 3$. Altogether, the $\mathrm{SO}(6,22)$ moduli of $\mathbb{T}^6$ join the $\mathrm{SO}(2,2)$ moduli of the non-perturbative $\mathbb{T}^2$ and the $2 \times (6+22)$ Wilson lines of the four-dimensional gauge fields and their duals, into the $\mathrm{SO}(8,24)$ coset.

To summarize, we have seen that the $\mathrm{SO}(8,24,\mathbb{Z})$ duality arises from a combination of the $\mathrm{SO}(7,23,\mathbb{Z})$ T-duality symmetry and the $\mathrm{Sl}(2,\mathbb{Z})$ electric-magnetic S-duality in 4 dimensions: it should therefore be an exact quantum symmetry of the three-dimensional theory.

2.2 Heterotic-type-II double triality

The heterotic string compactified on $\mathbb{T}^7$ and the type-II string compactified on $K_3 \times \mathbb{T}^3$ share the same supergravity action. Instead of going through the same procedure as above, we shall identify the $\mathrm{SO}(8,24)$ symmetry on the type-II side by using heterotic-type-II duality in six dimensions [20]. The identification of the moduli in $D = 6$ has been discussed in great detail in [1], where it was shown that the heterotic moduli are related to the type-II ones by SO(4,4) triality, mapping the vector representation to the conjugate spinor (see also [15]). At the $\mathbb{T}^4/\mathbb{Z}_2$ orbifold point with a square $\mathbb{T}^4$, this reduces to

$$R_1|_{\mathrm{H}} = \sqrt{R_1 R_2 R_3 R_4}_{|\mathrm{IIA}}\,, \qquad R_i|_{\mathrm{H}} = \sqrt{\frac{R_1 R_i}{R_j R_k}}_{|\mathrm{IIA}}\,, \qquad i,j,k = 2,3,4\,, \tag{2.7}$$

where the radii are measured in the respective string length units. These relations are supplemented by the identification of the string scale and six-dimensional string coupling,

$$l_{\mathrm{H}} = g_{6\mathrm{IIA}} l_{\mathrm{II}}\,, \quad g_{6\mathrm{IIA}} = \frac{1}{g_{6\mathrm{H}}}\,. \tag{2.8}$$

The three-torus on which both sides are further reduced is inert under heterotic-type-II duality, so we easily find from (2.3) the set of dilatonic moduli in the abelian part of the SO(8,24) coset representative,

$$x_{i=1,\ldots,8} = \left(\sqrt{R_1 R_2 R_3 R_4}, \sqrt{\frac{R_1 R_2}{R_3 R_4}}, \sqrt{\frac{R_1 R_3}{R_2 R_4}}, \sqrt{\frac{R_1 R_4}{R_2 R_3}}, \frac{R_5}{g_{6\mathrm{IIA}}}, \frac{R_6}{g_{6\mathrm{IIA}}}, \frac{R_7}{g_{6\mathrm{IIA}}}, \frac{R_5 R_6 R_7}{g_{6\mathrm{IIA}}} \right), \tag{2.9}$$

where all radii are measured in units of the type-II string length l_{II}.

The moduli appearing in (2.9) are not the usual $SO(4,4) \times SO(3,3)$ that arise from the reduction of the six-dimensional type-II theory on $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{T}^3$. The latter can however be reached by defining

$$\begin{aligned} y_1 &= \sqrt{x_1 x_2 x_3 x_4}\,, \qquad y_i = \sqrt{\frac{x_1 x_i}{x_j x_k}}\,, \qquad i,j,k = 2,3,4 \\ y_8 &= \sqrt{x_5 x_6 x_7 x_8}\,, \qquad y_i = \sqrt{\frac{x_8 x_i}{x_j x_k}}\,, \qquad i,j,k = 5,6,7\,, \end{aligned} \tag{2.10}$$

which indeed gives the more familiar parameterization

$$y_{i=1,\ldots,8} = \left(R_1, \ldots, R_7, \frac{1}{g_{3\mathrm{IIA}}^2} \right). \tag{2.11}$$

The type-IIA theory compactified on $K_3 \times \mathbb{T}^3$ therefore also appears to have a dynamically generated dimension, of size $\tilde{R}_8 = l_{\mathrm{II}}/g_{3\mathrm{IIA}}^2$. Again, this is not the same as the M-theory eleventh dimension of size $R_{11} = g_{\mathrm{II}} l_{\mathrm{II}}$, nor is it identical to the 8th radius $R_8 = l_{\mathrm{H}}/g_{3\mathrm{H}}^2 = V_3/l_{\mathrm{II}}^2$ that appeared naturally on the heterotic side. We see that the mapping of heterotic/$\mathbb{T}^4 \times \mathbb{T}^3$ to type II on $K_3 \times \mathbb{T}^3$ involves both an SO(4,4) triality on the $\mathbb{T}^4/\mathbb{Z}_2$ moduli, and another SO(4,4) triality on the non-perturbative $\mathbb{T}^4$ torus. This mapping amounts to a non-trivial embedding of the perturbative $SO(4,20) \times SO(3,3)$ type-II T-duality into the U-duality group SO(8,24). We have not worked out in detail the mapping of the Borel part of the moduli, but this can be easily done along the lines of [1].

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3. Exact F^4 threshold in $D=3$

Since $SO(8,24,\mathbb{Z})$ is believed to be an exact quantum symmetry of the heterotic string compactified on $\mathbb{T}^7$ or its dual versions, all amplitudes should be invariant under this duality group. In this section, we propose a U-duality invariant expression for F^4 amplitudes, which reproduces the known one-loop and tree-level answers on the heterotic and type-II sides respectively. By analyzing our proposal at weak coupling, we will be able to identify the instanton configurations that contribute, and extract their summation measure.

3.1 U-dual completion of perturbative amplitudes

Our starting point is the heterotic one-loop F^4 amplitude, which already incorporates most of the symmetries, namely the T-duality $SO(7,23,\mathbb{Z})$. It has been discussed in great detail in [1], building on previous work [21]. The result is

$$\Delta_{1-\text{loop}} = l_{\rm H}^5 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{p_R^4}{\bar\eta^{24}} Z_{7,23} t_8 F^4 \,, \tag{3.1}$$

where $Z_{7,23}$ is the Theta function of the perturbative heterotic compactification lattice. Here, p_R has modular weight $(0,1)$, and inserts the right-moving momentum corresponding to the choice of gauge boson F. Dualizing the vectors into scalars, we find

$$\Delta_{1-\text{loop}} = l_{\rm H} g_{3\rm H}^8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{p_R^4}{\bar\eta^{24}} Z_{7,23} t_4 (M_{7,23}\partial\phi)^4 \,, \tag{3.2}$$

where t_4 is the tensor t_8 with pairs of indices raised with the ϵ_3 antisymmetric tensor.

It is now simple to covariantize this result under U-duality. We propose that the exact $(\partial\phi)^4$ threshold in heterotic string on $\mathbb{T}^7$, or any of its dual formulations, is given by

$$I_{(\partial\phi)^4} = l_{\rm P} \int d^3x \sqrt{g} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{Z_{8,24}(g/l_{\rm H}^2, b, \phi, g_{3\rm H}^2)}{\bar\eta^{24}} t_8 [(e_{8,24}^{-1}\partial_\mu e_{8,24})_{ia} p_R^a]^4 \,, \tag{3.3}$$

where $l_{\rm P} = g_{3\rm H}^2 l_{\rm H}$ is the three-dimensional (U-duality invariant) Planck length. In this expression, $Z_{8,24}$ is the Theta function of the non-perturbative $\Gamma_{8,24}$ lattice specified by (2.4),

$$Z_{8,24}(\tau,\bar\tau) = (\tau_2)^4 \sum_{m_i,p_I,n^i} e^{-\pi\tau_2 \mathcal{M}^2 - 2\pi i \tau_1 v^t \eta_{8,24} v}$$
$$\mathcal{M}^2 = v^t M_{8,24} v \,, \quad v = (m_i, p_I, n^i) \,, \qquad i = 1,\ldots,8 \,, \quad I = 1,\ldots,16 \,. \tag{3.4}$$

where $M_{8,24} = e_{8,24}^t \, e_{8,24}$ is obtained from (2.4), and $\eta_{8,24}$ is the $SO(8,24)$ vielbein given in (2.5). The momenta, windings and gauge charges m_i, n^i, p^I are summed over the even self-dual lattice $\Gamma_{1,1}^8 \oplus D_{16}$. $Z_{8,24}$ is invariant under both U-duality

SO$(8,24,\mathbb{Z})$ and Sl$(2,\mathbb{Z})$ modular transformations of τ. $e_{8,24}^{-1}\partial_\mu e_{8,24}$ is the left-invariant one-form on the coset $[\mathrm{SO}(8)\times\mathrm{SO}(24)]\backslash\mathrm{SO}(8,24)$, pulled-back to space-time. It takes value in the $(\mathbf{8},\mathbf{24})$ component of the decomposition of the Lie algebra $so(8,24)$ under $so(8)\times so(24)$. Each index a in the $\mathbf{24}$ is contracted with an insertion p_R of the right-moving momenta (p_I,n^i) into the partition function $Z_{8,24}$. The four indices i in the $\mathbf{8}$ are contracted with the four momenta ∂_μ using the tensor t_8. This structure is also the one that arises in a one-loop four-scalar amplitude as shown in [22]. The conjecture (3.3) satisfies the following criteria:

(i) It is SO$(8,24,\mathbb{Z})$ invariant by construction;

(ii) It correctly reproduces the heterotic 1-loop coupling;

(iii) The non-perturbative contributions come from heterotic 5-branes and KK5-branes, which are the expected ones in $D=3$;

(iv) The result decompactifies to the known purely perturbative result in $D\geq 4$;

(v) Via heterotic/type-II and heterotic/type-I duality, the corresponding amplitude in type II and I shows the correct perturbative terms and the expected instanton corrections.

We shall now proceed to prove these claims. In order to avoid unnecessary complications, we will restrict ourselves to a particular subspace of moduli space, corresponding to the $\mathbb{T}^4/\mathbb{Z}_2\times\mathbb{T}^3$ orbifold point on the type-II side. It has been shown in [23, 1] that for this choice of Borel moduli, the Dedekind function $\bar\eta^{24}$ in (3.3) cancels against the action of p_R^4 on the D_{16} part of the lattice, and we are left with the simpler expression

$$\Delta_{(\partial\phi)^4}=l_{\mathrm{P}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}Z_{8,8}\left(\frac{g}{l_{\mathrm{H}}^2},b,g_{3\mathrm{H}}^2\right), \tag{3.5}$$

where the partition function $Z_{8,8}$ now runs over the lattice $\Gamma_{1,1}^8$ only.

3.2 The heterotic instanton expansion

To perform a weak coupling expansion of the result (3.5) we notice that defining $1/g_{3\mathrm{H}}^2=R_8/l_{\mathrm{H}}$ as in (2.2), the weak coupling expansion becomes a large R_8 expansion, so that we can adopt a lagrangian representation for the S^1 part and a hamiltonian representation for the remainder:

$$\Delta_{(\partial\phi)^4}=l_{\mathrm{P}}\frac{R_8}{l_{\mathrm{H}}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\sum_{p,q}\sum_{v}\exp\left(-\pi\frac{R_8^2|p-\tau q|^2}{l_{\mathrm{H}}^2\tau_2}+2\pi ip\;w_i n^i\right)\tau_2^{7/2}q^{p_L^2/2}\bar q^{p_R^2/2}\,, \tag{3.6}$$

where $v=(m_i,n^i)$ now denotes the $7+7$ perturbative momenta and windings. We apply the standard orbit decomposition method on the two integers (p,q), trading the sum over $\mathrm{Sl}(2,\mathbb{Z})$ images of (p,q) for a sum over images of the fundamental domain $\mathcal{F}$. The zero orbit gives back the perturbative result (3.1)

$$\Delta^{\text{zero}}_{(\partial\phi)^4} = l_{\mathrm{H}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{7,23}\,. \tag{3.7}$$

The degenerate orbit on the other hand, with representatives $(p,0)$, can be unfolded onto the strip $|\tau_1|<1/2$. The τ_1 integral then imposes the level matching condition $p_L^2 - p_R^2 = m_i n^i = 0$, and the τ_2 integral can be carried out in terms of Bessel functions to give

$$\Delta^{\text{deg}}_{(\partial\phi)^4} = 2 l_{\mathrm{H}} \sum_{p\neq 0}\sum_{v\neq 0} \delta(v^t \eta_{7,7} v) \left(\frac{p^2}{g_{3\mathrm{H}}^4 v^t M_{7,7} v}\right)^{5/4} K_{5/2}\left(\frac{2\pi}{g_{3\mathrm{H}}^2}|p|\sqrt{v^t M_{7,7} v}\right) e^{2\pi i p w_i n^i}\,. \tag{3.8}$$

From the expansion of the Bessel function

$$K_{5/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left[1 + \frac{3}{x} + \frac{3}{x^2}\right] \tag{3.9}$$

we see that these are non-perturbative contributions with classical action

$$\mathrm{Re}(S_{\text{cl}}) = \frac{2\pi}{g_{3\mathrm{H}}^2}|p|\sqrt{v^t M_{7,7} v}\,. \tag{3.10}$$

Choosing for v either "momentum" charges or "winding" charges, we find an action

$$\frac{1}{g_{3\mathrm{H}}^2}\frac{l_{\mathrm{H}}}{R_i} = \frac{V_6}{g_{\mathrm{H}}^2 l_{\mathrm{H}}^6}\,, \qquad \frac{1}{g_{3\mathrm{H}}^2}\frac{R_i}{l_{\mathrm{H}}} = \frac{V_6 R_i^2}{g_{\mathrm{H}}^2 l_{\mathrm{H}}^8}\,, \tag{3.11}$$

which identifies these instantons as heterotic 5 branes and KK5-branes respectively, wrapped on a $\mathbb{T}^6$ inside $\mathbb{T}^7$. The summation measure for these effects is easily extracted, and yields

$$\mu_{\mathrm{Het}}(N) = \sum_{d|N} \frac{1}{d^5}\,, \tag{3.12}$$

where $N = \gcd(p, m_i, n^i)$. We also note that the Bessel function $K_{5/2}$ in (3.9) exhibits only two subleading terms beyond the saddle point approximation, so that these instanton contributions do not receive any corrections beyond two loops.

Since NS5-brane instantons appear in the three-dimensional $(\partial\phi)^4$ result, one may wonder how they can not contribute in four-dimensions, where the F^4 threshold has been argued to be purely one-loop [24, 1]. Let us therefore investigate the decompactification of our result to four dimensions, using the parameterization described at the end of subsection 2.1. We thus decompose $Z_{8,24} = Z_{2,2} Z_{6,6}$ where $Z_{2,2}$ stands for the lattice sum on the non-perturbative two torus in the (7,8) direction and $Z_{6,22}$ is the lattice sum for the perturbative states in $D=4$. From (2.6) we see that decompactifying to 4D corresponds to a large volume limit of the two-torus, so that we use a lagrangian representation for $Z_{2,2}$ and hamiltonian for $Z_{6,6}$,

$$\Delta_{(\partial\phi)^4} = l_{\rm P}\frac{R_7 R_8}{l_{\rm H}^2}\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\sum_{p^\alpha,q^\alpha}\sum_{v} e^{-\frac{\pi}{\tau_2}(p^\alpha+\tau q^\alpha)(g_{\alpha\beta}+B_{\alpha\beta})(p^\beta+\bar\tau q^\beta)+2\pi i p^\alpha w_{\alpha j} n^j}\,\tau_2^3 q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}\,, \tag{3.13}$$

where $\alpha = 7,8$ and v are the perturbative charges in $D = 4$. We can now use the orbit decomposition for the sum over p, q. Then, the trivial orbit gives

$$\Delta^{{\rm zero};D=4}_{(\partial\phi)^4} = R_7\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2} Z_{6,6} = \frac{R_7}{l_{\rm H}^4}\Delta^{\rm 1-loop,4D}_{F^4}\,, \tag{3.14}$$

which decompactifies to the one-loop result in 4D. The degenerate orbit has $q = 0$, so that the τ_1 integral imposes the level-matching condition and the τ_2 integral gives

$$\Delta^{{\rm deg};D=4}_{(\partial\phi)^4} = R_7\sum_{p\neq 0}\sum_{v\neq 0}\delta(v^t\eta_{6,6}v)\frac{|p^t g p|}{|v^t M_{6,6} v|}K_2\left(2\pi\sqrt{|(p^t g p)(v^t M_{6,6} v)|}\right)e^{2\pi i p^t w n}\,. \tag{3.15}$$

Using the metric on the (7,8) torus

$$g = e_2^t e_2\,,\qquad e_2 = \frac{R_7}{l_{\rm H}}\begin{pmatrix}1 & \tau_1\\ 0 & \tau_2\end{pmatrix} \tag{3.16}$$

with τ defined in (2.6), one finds $\sqrt{p^t g p} = \frac{R_7}{l_{\rm H}}|p_1+\tau p_2|$. It is then obvious from the asymptotic form of the Bessel function that the term (3.15) disappears in 4D. The non-degenerate orbit vanishes similarly in the limit $R_7\to\infty$. We thus conclude that the conjecture (3.3) correctly reduces to the perturbative result in $D\geq 4$.

3.3 The type-II instanton expansion

We now consider the type-II interpretation of our conjecture (3.3). From the moduli identification (2.9), we see that the weak coupling expansion on the type-II side corresponds to the large volume expansion of the non-perturbative $\mathbb{T}^4$ in the 5–8 directions, with volume

$$v_4 = \frac{V_3^2}{g_{6{\rm IIA}}^4 l_{\rm II}^6}\,, \tag{3.17}$$

where $V_3 = R_5 R_6 R_7$. Using the by now familiar method, we decompose $Z_{8,8} = Z_{4,4}^{(1-4)} Z_{4,4}^{(5-8)}$ with hamiltonian and lagrangian representation for the two factors respectively, so that

$$\Delta_{(\partial\phi)^4} = l_{\rm P} v_4\int_{\mathcal F}\frac{d^2\tau}{\tau_2^2}\sum_{p^\alpha,q^\alpha}\sum_{m_i,n^i} e^{-\frac{\pi}{\tau_2}(p^\alpha+\tau q^\alpha)(g_{\alpha\beta}+B_{\alpha\beta})(p^\beta+\bar\tau q^\beta)+2\pi i p^\alpha w_{\alpha j} n^j}\,\tau_2^2 q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}\,, \tag{3.18}$$

where $\alpha = 5,6,7,8$ and $i = 1,\dots,4$. For use below, we define the rescaled metric G_4 on the non-perturbative four-torus

$$G_4 = g_{6\mathrm{IIA}}^2 g_4 \equiv e_4^t e_4 \,, \qquad e_4 = \mathrm{diag}\left(\frac{R_I}{l_{\mathrm{H}}}, \frac{V_3}{l_{\mathrm{H}}^3}\right), \qquad I = 5,6,7\,, \tag{3.19}$$

which depends on the geometric moduli only.

We can now perform in (3.18) the orbit decomposition for the sum over p, q (see [25, 18] for a discussion of the orbit decomposition for $Z_{d,d}$, $d > 2$, which generalizes the $d = 2$ case [26].) Then, the trivial orbit gives

$$\Delta^{\mathrm{zero}}_{(\partial\phi)^4} = g_{3\mathrm{IIA}}^2 l_{\mathrm{II}} \frac{V_3^2}{g_{6\mathrm{IIA}}^4} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} Z_{4,4} = \frac{V_3}{(g_{6\mathrm{IIA}} l_{\mathrm{II}})^4} \Delta^{\mathrm{tree,6D}}_{F^4}\,, \tag{3.20}$$

which, using (2.8) shows the correct 3D tree-level result, directly induced from the 6D tree-level F^4 coupling [1]. The degenerate orbit, with $q^\alpha = 0$, can be unfolded onto the strip $|\tau_1| < 1/2$. The τ_1 integral then imposes the level matching condition $p_L^2 - p_R^2 = 2 m_i n^i = 0$, and the τ_2 integral can be carried out in terms of Bessel functions to give

$$\begin{aligned}\Delta^{\mathrm{deg}}_{(\partial\phi)^4} = 2\frac{V_3}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} \sum_{p^i \neq 0} \sum_{(m_i,n^i)\neq 0} \delta(m_i n^i) \frac{1}{g_{6\mathrm{IIA}}} \frac{\sqrt{|p^t G_4 p|}}{\sqrt{m^t M_{4,4} m}} \times \\ \times K_1\left(\frac{2\pi}{g_{6\mathrm{IIA}}} \sqrt{p^t G_4 p} \sqrt{m^t M_{4,4} m}\right) e^{2\pi i p^\alpha w_{\alpha i} n^i}\,. \end{aligned}\tag{3.21}$$

From the argument of the Bessel function K_1, we recognize the contributions of euclidean D-branes wrapped on an even cycle of K_3, times a one-cycle of $\mathbb{T}^3$ (for p in the $5,6,7$ directions of the non-perturbative torus), or the whole $\mathbb{T}^3$ for p in the 8th direction. The latter case corresponds to the contributions of euclidean D6-branes, which start contributing in three dimensions.

For the non-degenerate orbit, the integral is dominated by the saddle point

$$\begin{aligned} q^\alpha g_{\alpha\beta}(p^\beta - \tau_1 q^\beta) + i\tau_2 m_i n^i = 0\,, \\ -(p^\alpha - \tau_1 q^\alpha) g_{\alpha\beta}(p^\beta - \tau_1 q^\beta) + \tau_2^2 (q^\alpha g_{\alpha\beta} q^\beta + m^t M_{4,4} m) = 0\,, \end{aligned}\tag{3.22}$$

as in our previous work [1, section 5.5]. This saddle point gives new non-perturbative contributions

$$\Delta^{\mathrm{n.d.}}_{(\partial\phi)^4} = \frac{4V_3}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} \sum_{p^i,q^i} \sum_{m_i,n^i} \left(\frac{(q^2)^2 + q^2 m^t M_{4,4} m + (m_i n^i)^2}{p^2 q^2 - (pq)^2}\right)^{3/4} K_{3/2}\left(\mathrm{Re}\, S_{\mathrm{cl}}\right) e^{i \Im S_{\mathrm{cl}}}\,. \tag{3.23}$$

to the F^4 threshold in type II, with classical action

$$\mathrm{Re}\, S_{\mathrm{cl}} = 2\pi \sqrt{\frac{p^2 q^2 - (pq)^2}{(q^2)^2} \left(\frac{(q^2)^2}{g_{6\mathrm{IIA}}^4 l_{\mathrm{II}}^4} + \frac{q^2 m^t M_{4,4} m}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} + (m_i n^i)^2\right)}\,. \tag{3.24}$$

In this expression, all inner products of p, q vectors are taken with the metric G_4 defined in (3.19). In particular, setting $m_i = n^i = 0$ and switching on one charge at a time for simplicity, we identify these non-perturbative effects as coming from NS5-brane instantons, with action

$$\frac{R_I R_J}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^2} = \frac{V_{K_3} R_I R_J}{g_{\mathrm{II}}^2 l_{\mathrm{II}}^6} \tag{3.25}$$

and KK5-brane instantons, with action

$$\frac{V_3 R_K}{g_{6\mathrm{IIA}}^2 l_{\mathrm{II}}^4} = \frac{V_{K_3} R_I R_J R_K^2}{g_{\mathrm{II}}^2 l_{\mathrm{II}}^8} \, . \tag{3.26}$$

The former already occur in four dimensions, but the latter are genuine three-dimensional effects. For general values of the charges p, q, m, we obtain contributions from boundstates of NS5 and KK5-branes with D-branes. It is a simple matter to derive the summation measure for N KK5-brane instantons in type-IIA$/K_3 \times \mathbb{T}^3$,

$$\mu_{\mathrm{IIA}}(N) = \sum_{d|N} \frac{1}{d^3} \, , \tag{3.27}$$

which is identical to the one derived for NS5-brane instantons in on IIA$/K_3 \times \mathbb{T}^2$ [1].

4. Discussion

In this work, we have proposed a novel exact amplitude for three-dimensional string theories with 16 supersymmetries, by completing the one-loop heterotic F^4 threshold into a U-duality invariant result. Our result generalizes the field theory analysis of [27], who found the exact F^4 amplitude in three-dimensional SU(2) Yang-Mills theory with 16 supersymmetries, to a full string theory context. In contrast to this work, we have not rigorously proven our proposal, for lack of knowledge of both supersymmetry constraints in 3d supergravity and harmonic analysis on grassmannians. Many consistency checks support our claim, however, including the agreement with the type-II tree-level result, the $D = 4$ decompactification limit and sensible instanton expansions.

Our result is particularly interesting on the heterotic side, where such examples are scarce and non-perturbative effects little understood. In addition to the heterotic five-brane instantons already found in [28] and the dual type-I D5-brane instantons [29], we have exhibited the contributions of KK5-instantons, which come into play for compactifications on 7-manifolds with a U(1) isometry. On the type-II side, we have recovered the familiar D-brane and NS5-instantons, together with the D6-brane and KK5-monopoles peculiar to three-dimensional compactifications. Although we have hardly mentioned it, the type-I picture is very similar to the

	IIA/$K_3 \times \mathbb{T}^n$	Het/$\mathbb{T}^n$	I/$\mathbb{T}^n$
Dp	2 (F^4)		D1: 0 (F^4) D5: 1 (R^2), 5 (F^4)
NS5	3 (F^4)	1 (R^2), 5 (F^4)	
KK5	3 (F^4)	5 (F^4)	5 (F^4)

Table 1: Instanton contributions to F^4 and R^2 couplings in theories with 16 supersymmetries. The entry denotes the exponent r appearing in the summation measure $\sum_{d|N} d^{-r}$.

heterotic one: Using heterotic/type-I duality [30], the heterotic one-loop amplitude reproduces the familiar disk and cylinder amplitudes, together with D1-instanton effects. The heterotic 5-brane and KK5-brane contributions turn into type-I D5-branes and KK5-branes, and their summation measure is unaffected by the duality. The table 1 summarizes the instanton summation measures for all cases known so far. It is a challenging problem to rederive the measures for the NS5-brane and KK5-instantons. A not lesser challenge will be to confront the long-eluded problem of two-dimensional supergravities with their formidable affine symmetry.

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